CSP FOR BINARY CONSERVATIVE RELATIONAL STRUCTURES

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Abstract. We prove that whenever $\mathbb{A}$ is a 3-conservative relational structure with only binary and unary relations then the algebra of polymorphisms of $\mathbb{A}$ either has no Taylor operation (i.e. CSP($\mathbb{A}$) is NP-complete), or generates an SD($\wedge$) variety (i.e. CSP($\mathbb{A}$) has bounded width).

1. Introduction

In the last decade, the study of the complexity of the Constraint Satisfaction Problem (CSP) has seen several major results due to universal algebraic methods (see e.g. [4], [12] and [2]).

Our aim in this paper is to study clones of polymorphisms of finite 3-conservative relational structures where the arity of all relations is at most two (ie. binary 3-conservative relational structures). We show that whenever such a structure admits a Taylor operation, its operational clone actually generates an SD($\wedge$) variety. In other words, we are going to show that if CSP of a binary conservative relational structure is not NP-complete then it must be solvable by a rather simple algorithm, local consistency checking (also known as the bounded width algorithm).

There have been numerous papers published on the behavior of conservative relational structures. We have been mostly building on three previous results: First, Andrei Bulatov proved in [5] the dichotomy of CSP complexity for conservative relational structures. Libor Barto recently offered a simpler proof in [1]. Meanwhile, Pavol Hell and Arash Rafiey have obtained a combinatorial characterization of all the tractable conservative digraphs in [8] and have observed that all tractable digraphs must have bounded width.

2. Preliminaries

A *relational structure* $\mathbb{A}$ is any set A together with a family of relations $\mathcal{R} = \{R_i : i \in I\}$ where $R_i \subset A^{n_i}$. We will call the number n_i the *arity* of R_i. As usual, we will consider only finite structures (and finitary relations) in this paper.

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Let R be an m-ary relation and $f : A^n \to A$ an n-ary operation. We say that f preserves R if whenever we have elements $a_{ij} \in A$ such that

$$\begin{gathered}(a_{11}, a_{12}, \dots, a_{1m}) \in R \\ (a_{21}, a_{22}, \dots, a_{2m}) \in R \\ \vdots \\ (a_{n1}, a_{n2}, \dots, a_{nm}) \in R\end{gathered}$$

then we also have

$$(f(a_{11}, \dots, a_{n1}), \dots, f(a_{1m}, \dots, a_{nm})) \in R.$$

If $\mathcal{R}$ is a set of relations then we denote by $\mathrm{Pol}(\mathcal{R})$ the set of all operations on A that preserve all $R \in \mathcal{R}$. On the other hand, if Γ is a set of operations on A then we denote by $\mathrm{Inv}(\Gamma)$ the set of all relations that are preserved by each operation $f \in \Gamma$.

One of the most important notions in CSP is primitive positive definition. If we have relations $R_1, \dots, R_k$ on A then a relation S on A is primitively positively defined using $R_1, \dots, R_k$ (pp-defined for short) if there exists a logical formula defining S that uses only conjunction, existential quantification, symbols for variables and predicates $R_1, \dots, R_k$ and "=".

Observe that the set $\mathrm{Pol}(\mathcal{R})$ is closed under composition and contains all the projections, therefore it is an *operational clone*. If $\mathbb{A} = (A, \mathcal{R})$ is a relational structure then $\mathrm{Inv}(\mathrm{Pol}(\mathcal{R}))$ consists of precisely all the relations that can be pp-defined using the relations from $\mathcal{R}$. We will call $\mathrm{Inv}(\mathrm{Pol}(\mathcal{R}))$ the *relational clone* of $\mathbb{A}$ (see [10] and [13] for a more detailed discussion).

Given an algebra $\mathbb{A}$, an instance of the *Constraint Satisfaction Problem* $\mathrm{CSP}(\mathbb{A})$ consists of a set of variables V and a set of constraints $\mathcal{C}$ where each constraint $C = (S, R)$ has a *scope* $S \subset V$ and a *relation* $R \subset A^S$ such that R belongs to the relational clone of $\mathbb{A}$. A *solution* of this instance is any mapping $f : V \to A$ such that $f_{|S} \in R$ for each constraint $(S, R) \in \mathcal{C}$. In this paper, we will only consider CSP instances where all relations have scopes of size at most two.

Let A be an algebra. We say that the variety generated by A is congruence meet semidistributive (SD($\wedge$) for short) if for any algebra B in the variety generated by A and any congruences α, β, γ in B we have

$$\alpha \wedge \beta = \alpha \wedge \gamma \Rightarrow \alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta.$$

We can draw any CSP instance as a constraint network (also known as potato diagram in algebraic circles): For each variable x we have the potato $B_x \subset A$ equal to the intersection of all the unary constraints on x and for each constraint of arity two we draw lines joining the corresponding potatoes. To solve the instance now means to choose in each potato B_x a vertex b_x so that whenever $\mathcal{C} = (\{x, y\}, R)$ is a constraint, the line from b_x to b_y corresponds to an element of R (see Figure 1 for an example).

We can express any primitive positive definition by a CSP instance; see Figure 2 for an example of such an expression.

Given $\mathbb{A}$, if the (2,3)-consistency checking algorithm as defined in [2] always returns a correct answer to any instance of $\mathrm{CSP}(\mathbb{A})$ (i.e. if there are no false positives) we say that $\mathbb{A}$ has *bounded width*. The following result shows a deep connection between bounded width and congruence meet semidistributivity.

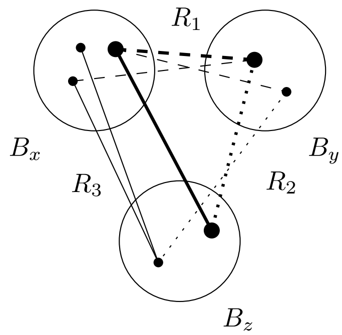

FIGURE 1. An example of a potato diagram with three variables and three binary relations (solution in bold)

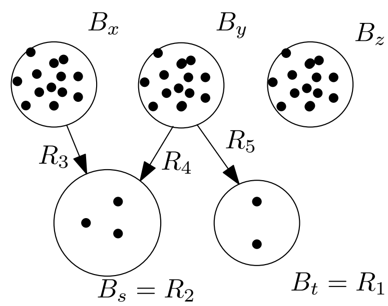

FIGURE 2. The potato diagram for the ternary relation $S = \{(x,y,z) : \exists s, \exists t,\ s \in R_1 \wedge t \in R_2 \wedge (x,s) \in R_3 \wedge (y,s) \in R_4 \wedge (y,t) \in R_5\}$.

Theorem 1. *Let $\mathbb{A}$ be a finite relational structure containing all the unary relations. Then the following are equivalent:*

(1) $\mathbb{A}$ has bounded width,
(2) the variety generated by $\mathrm{Pol}(\mathbb{A})$ is congruence meet semidistributive.
(3) $\mathrm{Pol}(\mathbb{A})$ contains 3ary and 4ary weak near unanimity (WNU) operations with the same polymer, i.e. there exist $u, v \in \mathrm{Pol}(\mathbb{A})$ such that for all $x, y \in A$ we have:

$$u(x,x,y) = u(x,y,x) = u(y,x,x) = v(y,x,x,x) = \cdots = v(x,x,x,y)$$

Proof. For "1 ⇒ 3" see the upcoming survey [7], while "2 ⇒ 1" is the main result of [3].

To prove "3 ⇒ 2", it is enough to observe that the equations for idempotent 3ary and 4ary WNU operations with the same polymer fail in any nontrivial variety of modules and therefore, as shown in [9, Theorem 9.10], the third condition implies congruence meet semidistributivity (this is true even in the case of infinite algebras, as shown in [11]). □

We say that an n-ary operation t is Taylor if it satisfies for every $1 \leq k \leq n$ some equation of the form $t(u_1, \ldots, u_n) \approx t(v_1, \ldots, v_n)$ where u_k is the variable x and v_k a different variable y. Obviously, if an operational clone contains only projections,

it can never have a Taylor operation. Moreover, if $\mathbb{A}$ is a relational structure with all the constants (one-element unary relations) such that $\mathrm{Pol}(\mathbb{A})$ does not have any Taylor operation then $\mathrm{CSP}(\mathbb{A})$ is known to be NP-complete (see [4, Corollary 7.3] together with [9, Lemma 9.4]).

A relational structure $\mathbb{A}$ is *conservative* if $\mathbb{A}$ contains all the possible unary relations on A. We will call a relational structure $\mathbb{A}$ *3-conservative* if $\mathbb{A}$ contains all the one, two and three-element unary relations. We will extend the usual definition of a digraph to include any relational structure with precisely one binary relation and no relations of greater arity (i.e. without limitation on the number of unary relations).

3. Red, yellow and blue pairs

Assume that $\mathbb{A}$ is a conservative relational structure such that $\mathrm{Pol}(\mathbb{A})$ contains a Taylor operation. Then for every pair of vertices $a, b \in A$ there must exist a polymorphism of $\mathbb{A}$ that, when restricted to $\{a, b\}$, is a semilattice, majority or minority. If there was a pair without such a polymorphism then a result by Schaefer [14] gives us that the operations in $\mathrm{Pol}(\mathbb{A})$ restricted to $\{a, b\}$ are projections and so $\mathrm{Pol}(\mathbb{A})$ can not have a Taylor term.

We will color each pair $\{a, b\} \subset A$ as follows:

(1) If there exists $f \in \mathrm{Pol}(\mathbb{A})$ semilattice on $\{a, b\}$ we color $\{a, b\}$ *red*, else
(2) if there exists $g \in \mathrm{Pol}(\mathbb{A})$ majority on $\{a, b\}$ we color $\{a, b\}$ *yellow*, else
(3) if there exists $h \in \mathrm{Pol}(\mathbb{A})$ minority operation on $\{a, b\}$ we color $\{a, b\}$ *blue*.

In [5], Andrei Bulatov proves the Three Operations Proposition (we change the notation to be compatible with ours and omit the last part of the proposition which we will not need):

Theorem 2. *There are polymorphisms $f(x, y), g(x, y, z), h(x, y, z) \in \mathrm{Pol}(G)$ such that for every two-element subset $B \subset V(G)$:*

- *$f_{|B}$ is a semilattice operation whenever B is red, and $f_{|B}(x, y) = x$ otherwise,*
- *$g_{|B}$ is a majority operation if B is yellow, $g_{|B}(x, y, z) = x$ if B is blue, and $g_{|B}(x, y, z) = f_{|B}(f_{|B}(x, y), z)$ if B is red*
- *$h_{|B}$ is a minority operation if B is blue, $g_{|B}(x, y, z) = x$ if B is yellow, and $g_{|B}(x, y, z) = f_{|B}(f_{|B}(x, y), z)$ if B is red.*

While we omit the proof here, it is actually quite elementary; one needs only to keep composing terms.

Observation 3. *If G is such that all its pairs are red or yellow, then G has bounded width since the operations*

$$u(x, y, z) = g(f(f(x, y), z), f(f(y, z), x), f(f(z, x), y))$$
$$v(x, y, z, t) = g(f(f(f(x, y), z), t), f(f(f(y, z), x), t)), f(f(f(z, x), y), t))$$

are a pair of 3-ary and 4-ary WNUs with the same polymer. If x, y are red then $u(x, x, y) = v(x, x, x, y) = f(x, y)$ and if x, y are yellow then $u(x, x, y) = v(x, x, x, y) = x$.

Proof. Straightforward verification and case analysis. □

By Theorem 1, it is enough to show that if $\mathrm{CSP}(G)$ is tractable then G does not have any blue pair of vertices.

At this point we could end our paper and refer the reader to the article [8] which, among other things, shows by combinatorial methods that if $\mathrm{CSP}(G)$ is not NP-complete, then all pairs of vertices of G are either yellow or red. However, we would like to present a short algebraic proof of this statement.

4. Conservative digraphs

We first show our result for conservative digraphs then generalize it. We proceed by contradiction. Let us for the remainder of this section fix a digraph G that is a vertex-minimal counterexample, i.e. $\mathrm{CSP}(G)$ is tractable, yet there exists a blue pair $\{a, b\} \subset V(G)$.

Proposition 4. *The set* $\mathrm{Inv}(G)$ *contains the relation* R *consisting of the triples* $(b, b, b), (a, a, b), (a, b, a), (b, a, a)$.

Proof. Consider the ternary relation

$$R = \{(t(a, a, b), t(a, b, a), t(b, a, a)) : t \in \mathrm{Pol}_3(G)\},$$

where $\mathrm{Pol}_3(G)$ denotes all the ternary polymorphisms of G. A little thought gives us that $R \in \mathrm{Inv}(G)$. Since the projections π_1, π_2, π_3 and the minority h belong to $\mathrm{Pol}_3(G)$, substituting these polymorphisms for t yields that $(a, a, b), (a, b, a), (b, a, a)$ and (b, b, b) lie in R.

Assume now that R contains some other element. If $(a, a, a) \in R$ then there exists $t \in \mathrm{Pol}_3(G)$ that acts as a majority on $\{a, b\}$ and a, b should have been yellow. If $(b, b, a) \in R$ then there exists some t such that

$$t(a, a, b) = t(a, b, a) = b$$
$$t(b, a, a) = a.$$

Since $\{a, b\}$ is not red, $t(b, b, a) = t(b, a, b) = a$ and $t(a, b, b) = b$ (otherwise we could define a semilattice operation $t(x, x, y)$ or $t(x, y, x)$ or $t(y, x, x)$). But then $f(x, y, z) = h(t(x, y, z), t(y, z, x), t(z, x, y))$ is a majority operation on $\{a, b\}$, a contradiction.

The cases $(b, a, b), (a, b, b) \in R$ are symmetric. $\square$

We need to introduce a bit of notation at this point: Let s be a solution of a CSP instance I. If the value of the variable x in the solution s is equal to c then we say that *s passes through c at x*, otherwise we say that *s avoids c at x*. If (x, y, z) is a triple of variables then we say that I realizes the triple (c, d, e) at (x, y, z) if there exists a solution s of I such that s passes through c at x, through d at y and through e at z. We will often leave out the "at x" or "at (x, y, z)" part if the location is clear from the context.

Since $R \in \mathrm{Inv}(G)$, there is a $\mathrm{CSP}(\mathbb{A})$ instance I and three variables x, y, z in I such that I realizes precisely all the triples in R at (x, y, z). Denote by P_x, P_y, P_z the "variable potatoes" and let $\{B_j : j \in J\}$ be the set of all the other potatoes in the constraint network of I. We can choose I so that the sum of the sizes of its potatoes is minimal among all possible $\mathrm{CSP}(\mathbb{A})$ instances realizing R. (Recall that we have defined $\mathrm{CSP}(\mathbb{A})$ instances so that the constraints of I can use any unary and binary relations from $\mathrm{Inv}(G)$ but no relations of greater arity.)

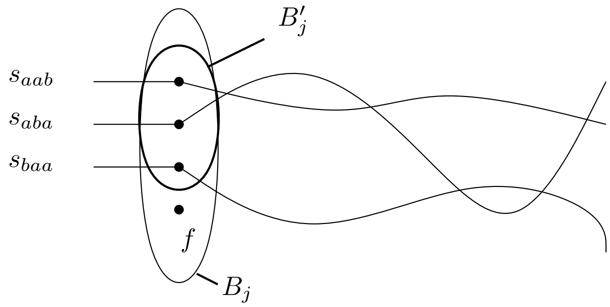

FIGURE 3. Proof of Observation 5. The original set B_j shrinks to B'_j after we forbid the element f

Observe that if r, s, t are two solutions of I and p is a ternary operation preserving all the relations used in I (because, say, $p \in \operatorname{Pol}(\mathbb{A})$) then $p(r, s, t)$ is also a solution of I.

Observation 5. *For every $j \in J$ we have $|B_j| \leq 3$ and $|B_j| \neq 1$.*

Proof. Assume that there is a set B_j with at least four distinct elements. Let $s_{aab}, s_{aba}, s_{baa}$ be some solutions of I realizing (a, a, b), (a, b, a), (b, a, a) at (x, y, z).

Now let B'_j be the subset of vertices of B_j that are being passed through by some of the solutions $s_{aab}, s_{aba}, s_{baa}$. Obviously, $|B'_j| \leq 3$, so if we replace B_j by B'_j (using a unary constraint), we get a smaller instance I'. Since $s_{aab}, s_{aba}, s_{baa}$ and $h(s_{aab}, s_{aba}, s_{baa})$ (where h is the polymorphism from Theorem 2) are solutions of I' and any solution of I' is also a solution of I, the instance I' realizes precisely all the elements of R at (x, y, z), a contradiction with the minimality of I.

If some B_j was a singleton, we could just remove B_j from I, replace constraints with scopes of the form $\{i, j\}$ by their projections, and obtain a smaller instance realizing R. □

Observation 6. *The pair $\{c, d\}$ is blue for every $c, d \in B_j$ with $j \in J$*

Proof. A kind of absorption is at work here.

Assume first that $\{c, d\}$ is red and $f(c, d) = d$ where f is the polymorphism from Theorem 2. We know that there exists a solution s of I that passes through d (otherwise, we could just delete d). If now r is a solution that passes through c then $f(r, s)$ is a solution that passes through d. What is more, since $f = \pi_1$ on the potatoes P_1, P_2, P_3, the solution $f(r, s)$ realizes the same triple as r. Therefore, we can remove c from B_j without affecting the set of triples realized at (x, y, z).

The situation for $\{c, d\}$ being yellow is similar. Again, let s be a solution passing through d and r be a solution that passes through c. Then $g(r, s, s)$ is a solution that passes through d and realizes the same triple as r, allowing us to eliminate c from B_j. □

Observation 7. *Assume $|B_j| = 3$. Then for every $c \in B_j$ there is exactly one of $(a, a, b), (a, b, a), (b, a, a)$ whose all realizations pass through c.*

Proof. As in the proof of Observation 5 let $s_{aab}, s_{aba}, s_{baa}$ be some solutions realizing (a, a, b), (a, b, a), (b, a, a).

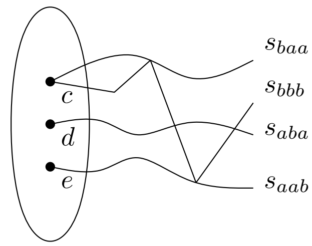

FIGURE 4. Proving Observation 8

Assume that there is a c through which passes neither s_{aab}, nor s_{aba} nor s_{baa}. Then by eliminating c from B_j we get a smaller instance and a contradiction like in the proof of Observation 5.

Assume now that there exists a $c \in B_j$ such that both s_{aab} and s_{aba} pass through c at j. Let d be the element of B_j through which passes the solution s_{baa}. If we now replace B_j with $\{c, d\}$ we, again obtain a smaller instance realizing R. □

Observation 8. *For every $j \in J$ we have $|B_j| = 2$.*

Proof. Let us assume that $B_j = \{c, d, e\}$ for some j. We already know that all the pairs in B_j must be blue. Let $s_{aab}, s_{aba}, s_{baa}$ and s_{bbb} be some solutions realizing (a, a, b), (a, b, a), (b, a, a) and (b, b, b).

Using Observation 7, we can assume that s_{baa} passes thought c, s_{aba} passes thought d and s_{baa} passes thought e (see Figure 4). Without loss of generality assume that s_{bbb} passes through c. Now consider the solution $h(s_{baa}, s_{aba}, s_{bbb})$ where h again comes from Theorem 2. Since everything is blue, this is a realization of (a, a, b) that passes through $h(c, d, c) = d$. But then by Observation 7 we have $d = e$, a contradiction. □

Let us now look at our instance: We have binary potatoes and subdirect binary relations (for otherwise we could make I smaller) between them. But the binary relations have to be h-invariant and h acts as a minority on all the potatoes. This leaves us very little room.

Observation 9. *Let $T \leq_s A \times B$ where $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ are binary algebras with a common minority operation h. Then either $T = A \times B$ or $T = \{(a_1, b_1), (a_2, b_2)\}$, or $T = \{(a_1, b_2), (a_2, b_1)\}$.*

Proof. Since the relation T is subdirect, any counterexample has to look like the letter "N", say (without loss of generality) $T = \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\}$. But then we must have in T the pair $(h(a_1, a_1, a_2), h(b_1, b_2, b_2)) = (a_2, b_1)$, so T is actually the full relation. □

We now see that the binary relations of I are all invariant under the majority polymorphism and so I must realize the triple

$$m((a, a, b), (a, b, a), (b, a, a)) = (a, a, a),$$

a contradiction.

Having proved the main result, we can now examine the proof and see that our arguments work verbatim in an even more general case.

Corollary 10. *If $\mathbb{A}$ is a finite 3-conservative relational structure whose all relations are binary or unary and* $\mathrm{Pol}(\mathbb{A})$ *admits a Taylor term then the variety generated by* $\mathrm{Pol}(\mathbb{A})$ *is* $\mathrm{SD}(\wedge)$.

Restating our result in the CSP complexity setting, we obtain:

Corollary 11 (Dichotomy for 3-conservative CSPs with binary relations)**.** *If $\mathbb{A} = (A, R_1, \dots, R_k)$ is a finite 3-conservative relational structure whose all relations are binary or unary and* $\mathrm{CSP}(\mathbb{A})$ *is tractable then* $\mathrm{CSP}(\mathbb{A})$ *has bounded width.*

Note, however, that our result can not be generalized to relational structures with ternary relations. For example the structure $\mathbb{A}$ on $\{0,1\}$ with all the unary relations plus the relation $S = \{(x,y,z) : x+y+z = 0 \pmod 2\}$ has $\mathrm{Pol}(\mathbb{A})$ consisting of all the idempotent linear mappings over $\mathbb{Z}_2$, so $\mathrm{Pol}(\mathbb{A})$ can not contain any WNUs of even arity.

Our result is also false for 2-conservative relational structures with arity of all relations at most two. Let $\mathbb{A}$ be the relational structure on $\{0,1\}^2$ with all unary relations of size one and two and three equivalence relations α, β and γ. These three equivalences correspond to the following partitions of $\{0,1\}^2$:

$$\{\{(0,0),(1,1)\},\{(0,1),(1,0)\}\}$$
$$\{\{(0,0),(0,1)\},\{(1,0),(1,1)\}\}$$
$$\{\{(0,0),(1,0)\},\{(0,1),(1,1)\}\}$$

It is straightforward to check that $\mathbb{A}$ admits the idempotent Taylor term $p(x,y,z) = x+y+z$ where addition is taken componentwise and modulo 2. However, the variety generated by $(\{0,1\}^2, \mathrm{Pol}(\mathbb{A}))$ is definitely not $\mathrm{SD}(\wedge)$ since α, β, γ are congruences of $(\{0,1\}^2, \mathrm{Pol}(\mathbb{A}))$ such that $\alpha \wedge \beta = \alpha \wedge \gamma = 0_A$ and $\alpha \wedge (\beta \vee \gamma) = 1_A$.

5. Closing remarks

It is remarkably difficult to obtain a digraph that would be tractable, yet would not have bounded width (though such beasts do exist; see for example the construction from [6]). Our paper shows why – such digraph needs to allow some nonconservative ternary operation which are hard to come by.

As we have seen, our result is quite tight. However, some generalizations might still be possible. At the moment, we do not know if our result holds for 2-conservative digraphs and if Corollary 10 holds when we drop the finiteness condition. We suspect that the answer to both questions will be negative, but the counterexamples might turn out to be illuminating.

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